Modern Classical Spin Dynamics

Abstract: The near-century-old Stern-Gerlach experiment played an important role in the philosophy of quantum mechanics. 50 years ago Bell drew drastic conclusions about the nature of reality based on a model of Stern-Gerlach, yet most detailed analysis of spin in nonuniform magnetic fields has occurred post-Bell. Recent focus on *work* in non-equilibrium thermodynamics has potential significance for quantum mechanics so we develop a spin dynamical analysis of work in an inhomogeneous field. A *small angle approximation* analysis is performed. We derive a novel Stern-Gerlach gradient threshold relation and a decay rate for precession in a non-uniform field. The theory is compared to a quantum analysis and an experiment to test this theory is proposed.

Keywords: Classical spin, Quantum spin, Stern-Gerlach, Precession decay rate, Work in magnetic field, Dynamic spin stability

The Stern-Gerlach experiment, almost a century old, played an important role in the philosophy of quantum mechanics, serving as prototype for preparation of the quantum state and model for certain kinds of quantum measurements [1]. In fact, the entire subject of quantum mechanics can be developed using only the results of the SG experiment [2]. Yet there exists [3] no complete quantum theoretic treatment of the SG experiment. Half a century ago Bell concluded, on the basis of his model of Stern-Gerlach, that classical models cannot reproduce quantum mechanical statistical predictions [4]. Less than two decades ago Jarzynski developed an equality that relates work on a non-equilibrium thermodynamics system to the free energy of the system [5] [6]. Recently Deissler investigated the fundamental question of whether or not a magnetic field, as in Stern-Gerlach, does work on an atom [7]. The relevance of this is that "*work is not an observable*" in the standard sense [8]. Specifically, work is not represented by a Hermitian operator, and thus is not an ordinary quantum observable: the number of possible values of work $W = E_f - E_i$ is typically larger than the dimension of the space of states; hence a Hermitian operator representing work cannot exist [9]. If no work is performed, the quantum formulation of Stern-Gerlach is more straightforward, since, per the *Correspondence Principle*, a quantum Hamiltonian is derived by quantizing the classical Hamiltonian. But if the field *does* do work on the system, the nonexistence of a Hermitian work operator complicates this quantization process. This is relevant to the quantum mechanical model of the Stern-Gerlach experiment on which Bell's model is based, which does *not* consider such work.

This paper extends Deissler's analysis of work on a dipole in a magnetic field. I demonstrate that the issue is more complex than was assumed by Deissler in that work *may* or *may not* be performed on an atom in an inhomogeneous magnetic field. I establish a novel threshold that is relevant to Stern-Gerlach spin dynamics and also a novel decay rate of precession. Sec. I provides an historical analysis. Sec II develops an Energy-Exchange analysis. Sec III develops small angle approximation dynamics. Sec. IV derives trajectory info from an *endpoint*-based analysis. Sec. V calculates the precession decay rate, while Sec. VI analyzes dynamic spin stability. Sec. VII derives an asymmetric approximation for Stern-Gerlach and Sec. VIII reviews a recent edge-effect analysis of SG. Finally, Sec. IX is the summary, conclusion, and appendix.

Sec. I Historical Analysis

Prior to Bell (1964) the *only* analysis of *work* on a classical dipole in a uniform field was Goldstein's (1951) in which case no work is done [10]. Not until fifteen years *after* Bell's theorem was based on Stern-Gerlach did Coombes analyze an atom in a non-uniform field; his analysis of semi-classical (no spin) atoms found that any increase in translational kinetic energy comes at the expense of the internal energy of the atom [11]. Thus Bell was completely unaware of two major aspects of atomic physics that have recently become the subject of considerable interest: the exchange of energy between internal modes as free energy and its relation to work. Based on Goldstein and Coombes, *classically, a static magnetic field does no work*. Any increase in translational kinetic energy of the atom is associated with a decrease in the internal energy of the atom. However, when intrinsic spin of the atomic electron is considered, spin Hamiltonian H_s is $-\vec{\mu}\cdot\vec{B}$ with $\vec{\mu}$ the intrinsic magnetic moment of the electron, $\vec{B}$ the magnetic field. For potential energy $U=-\vec{\mu}\cdot\vec{B}$ force on the dipole is $\vec{F}=-\vec{\nabla}U$, so any change in H_s corresponds to work done by the magnetic field, $W=-\Delta H_s$. But work done on spin conflicts with the fact that no work is done on the orbital magnetic moment, and Deissler concludes that for the magnetic field to do no work on the spin contribution to the magnetic moment, the electron would need to have an intrinsic rotational kinetic energy associated with its spin. He notes that Dirac's analysis implies that rotational energy of spin is contained in the mc^2 energy.

In another classical treatment of Stern-Gerlach [27] Franca claims that the interaction between small circuits and paramagnetic molecules is such that

> *"There is an energy exchange between the coils of the conductor and the magnetic molecules close to these coils. A similar phenomenon occurs with the atoms which cross a strong Stern-Gerlach electromagnet."*

Yet Deissler's analysis of spinning charged classical dipoles concludes that no work is done on spin because any increase in translational kinetic energy of the atom is compensated by decrease in rotational spin energy. But quantum spin does not 'give up' rotational energy of motion in order to compensate changes in kinetic energy. Instead, [12][13], the change in translational kinetic energy experienced by a magnetic dipole in an inhomogeneous field is compensated by change in precessional rotational kinetic energy. This implies a *limit* to the compensated changes, beyond which a non-uniform magnetic field actually *does* perform work on the particle.

For an atom the magnetic field does no work on the electron-orbital contribution to the magnetic moment. The source of translational kinetic energy is the internal energy of the atom; the atom rearranges its orbital configuration to compensate for the change in external field. Deissler's conclusion (as to whether or not work is done on electron-spin contributions to the magnetic moment) depends upon whether the electron has an intrinsic rotational kinetic energy associated with a spin. This is based on analysis of classical charged spinning objects, a ball and a ring. In the ring, per Faraday's law of induction, an electromotive force is produced that in turn produces torque on the ring, resulting in a change of angular rotational frequency or rate of spin. Analysis of a charged spinning ball shows any increase in translational kinetic energy of the ball to be accompanied by a corresponding decrease in rotational kinetic energy associated with the change in the precession rate of the ball. Deissler observes that the direction of precession may be opposite to the spin direction, in which case an increase in the magnitude of precession rate can correspond to a decrease in rotational kinetic energy. So there are two contributions to rotational energy: rotation about the z'-body axis and precessional rotation about the z-axis. For a given spin rate $\dot{\phi}$ and precession angle θ a change from θ to $\pi - \theta$ (a "spin flip") corresponds to a change in rotational energy, even for a uniform magnetic field. This is so since direction of precession is the same for both states, while the projections of the rotation vectors about the figure axes onto the z-axis are opposite. This analysis, occurring 44 years after John Bell's theorem, has relevance to Bell's remark [4] (p.141) that classical analysis of spin splitting "*would require 'compass needles' pointing in the wrong direction. And anyway it is not dynamically sound.*" He further noted that this phenomenon "*made physicists despair of finding any consistent space-time picture of what goes on on the atomic and subatomic scale.*"

The relevant details needed to understand this apparent instability were unknown to Bell and were uncovered by Deissler. Yet Deissler's analysis appears incomplete, since for the magnetic field to do no work on the spin the electron would need an intrinsic rotational kinetic energy associated with its spin, interpreted as implied by Dirac's mc^2 energy. In order for a field to do no work on the particle, any change in the translational kinetic energy must be compensated for by change in the intrinsic rotational kinetic energy of the particle in analogy with classical spinning objects. This analysis may be compatible with classical charged spinning rings and balls, but if we assume that the magnitude of the intrinsic spin is quantized and no change in mc^2-rotational kinetic energy is brought about by an external magnetic field then this will conflict with Deissler's conclusion that no work is done on the spin; the internal energy of rotation is finite, whereas changes in external fields are effectively unlimited. Deissler's 'no work' conclusion is satisfied for precessing spin in an inhomogeneous field if any change in the translational kinetic energy is compensated for by change in rotational kinetic energy associated with the change in the precession frequency (or the angle of precession.) But the angle of precession is finite; once a dipole is aligned with the field, the angle is zero and no more change in precession energy can occur. At this point no more precession energy is available to compensate the change in translational kinetic energy. Yet force $\vec{F} = \vec{\nabla}(\vec{\mu} \cdot \vec{B})$ is *maximum* on an aligned dipole. Thus either change in intrinsic (mc^2) rotational kinetic energy occurs, or the change is uncompensated and work *is* done on the dipole.

Sec. II An Energy-Exchange Analysis

Although magnetic force on a charged particle is orthogonal to the path, and hence does no work, this is not the case for a magnetic dipole. How can the force of the field gradient acting on the magnetic dipole be exerted over a finite distance, $F \cdot x$, and yet do no work on the system (KE = kinetic energy of translation)? Normally

$$\int_0^x d(KE) = '\mathit{work}'. \qquad 1$$

The answer is based on the assumption that precession energy (PE) is exchanged with (changed into) kinetic energy (KE) as described in the following *Energy-Exchange theorem*. Assume that the particle is deflected from the *z*-axis (*x = 0*) to a distance *x* from the *z*-axis and that precession energy PE *decreases* to compensate the increase in kinetic energy of translation until the precession energy is exhausted at position *q*.

$$\int_0^x d(KE) + \int_0^q d(PE) \quad = \quad \sum \Delta E = '\mathit{work}' \qquad 2$$

To show the energy exchange explicitly we break the kinetic energy integral from 0 to *x* into two integrals, from 0 to *q* and from *q* to *x*:

$$\int_0^x d(KE) + \int_0^q d(PE) \quad = \quad \underbrace{\int_0^q d(KE) + \int_0^q d(PE)}_{NO\ \ WORK} \; + \; \underbrace{\int_q^x d(KE)}_{WORK} \;\equiv \Delta E \qquad 3$$

As work ($W = \Delta E$) is defined as the *change in energy*, we observe that no work is done from zero to *q* as the total energy change is zero. After the particle aligns at position *q* (where precession has ended) work is done from *q* to *x*.

In this model no work is done while the spin is in process of alignment with the local field, but, once aligned, the field performs work on the particle. The dynamics (which caused Bell to despair) is even more complicated than Deissler assumed as there are now *two* phases: a phase in which no work is done by the field, due to internal compensation, followed by a phase in which the field performs work on the particle.

In an SG-apparatus, a magnetic moment traverses a non-uniform magnetic field, experiencing a gradient-based deflecting force. Uniform magnetic fields do *not* do work on particles, and inhomogeneous magnetic fields do not do work on a *spin-less atom*, but analysis of a quasi-particle defined by electron spin, local magnetic field, and precession energy shows that spin models of particles in an inhomogeneous field will *exchange energy between local energy modes* to compensate for changes in the local field. We are thus motivated to describe the energy transfers between eigenmodes, in which case an *Energy-Exchange theorem* is basic [12][13]:

The Energy-Exchange theorem –

> *If a physical system possesses two energy modes, M_0, M_1, coupled to a common variable θ, and energies of the modes are not separated by a quantum gap $\Delta\varepsilon > 0$, then if the common variable changes, $d\theta/dt \neq 0$, the modes will exchange energy.*

Assume that total energy is $\varepsilon = \varepsilon_0 + \varepsilon_1$ when $H_i\,|\psi\rangle = \varepsilon_i\,|\psi\rangle$ and that total energy $H = H_0 + H_1$ is conserved:

$$\frac{dH}{dt}=0 \Rightarrow \frac{dH_0}{dt}+\frac{dH_1}{dt}=0 \Rightarrow \frac{dH_0}{d\theta}\frac{d\theta}{dt}+\frac{dH_1}{d\theta}\frac{d\theta}{dt}=0 \quad \Rightarrow \quad \left(\frac{dH_0}{d\theta}+\frac{dH_1}{d\theta}\right)\frac{d\theta}{dt}=0 \qquad 4$$

Since $d\theta/dt \neq 0$ then $$\frac{dH_0}{d\theta}=-\frac{dH_1}{d\theta} \qquad 5$$

and energy flows between mode M_0 and M_1. QED

We apply this principle to the Stern-Gerlach experiment as follows. A neutral silver atom enters the field with *z*-axis momentum and its intrinsic magnetic moment $\vec{\mu}$ at angle θ to the local B-field in the *x*-direction. The force of the *x*-directed field gradient $\vec{F}=\vec{\nabla}(\vec{\mu}\cdot\vec{B})$ accelerates the particle in the *x*-direction while making no change to the initial *z*-momentum. The change in kinetic energy due to particle acceleration is compensated by the precession energy in order to conserve local energy: $E_{in}=E_{out}$.

$$-\vec{\mu}\cdot\vec{B}=mv_x^2/2-|\vec{\mu}||\vec{B}| \qquad 6$$

where the *z*-momentum energy has been canceled for both sides. The change in translational kinetic energy, $mv_x^2/2$ is thus

$$mv_x^2/2=\mu B(1-\cos\theta)\,, \qquad 7$$

where $\mu=|\vec{\mu}|$, $B=|\vec{B}|$. Applying the constant acceleration formula, $v_f^2=v_i^2+2ax$, as an approximation, yields

$$m(ax)\cong \mu B(1-\cos\theta) \qquad 8$$

and hence an approximate θ-dependent component of deflection

$$x=\frac{\mu B}{ma}(1-\cos\theta)\,. \qquad 9$$

The distance $x\ [=x(1)]$ is the amount of deflection from the z-axis that a particle with initial angle of precession θ experiences when the particle becomes aligned with the local field. The relation $x=f(\theta)$ is such that the deflection x is determined by *initial* angle θ as well as field strength and field gradient. The *Energy-Exchange Principle* constrains local dynamics until locally available energy is exhausted. A quasi-particle with internal degrees of freedom traversing a non-uniform field locally conserves energy over its N-degrees-of-freedom. A precessing, translating, magnetic moment has two such degrees of freedom, but finite compensation mechanisms, when exhausted, can no longer accommodate changes in the local field. At such time the local gradient begins delivering power. So a local field gradient will not impart energy as work to the quasi-particle, either atom or precessing spin, *until* precession energy has been converted to deflection energy. If precession energy vanishes ($\theta=0$) the local gradient drives the translation of the particle. When the *precession-battery* is drained the field takes over. Navasques and Popescu perform a similar analysis for photons [14].

Sec. III A Small Angle Approximation

Consider the maximum deflection, $x(3)$, an incoming aligned particle $(\theta = 0)$ would experience. For an incoming moment with initial angle θ the precession energy will be exchanged until the moment is aligned at $z(1)$; deflection at this point is $x(1)$. From $z(1)$ to $z(2) = L$ the aligned particle experiences the maximum force of the field gradient, and is deflected to $x(2)$. Since $x(3) \geq x(2) \geq x(1)$ we examine the small angle approximation $x(2) \approx x(3) - x(1)$. From the distance formula we see that

$$v^2 = {v_x}^2 + 2ax = v_x^2 + 2\left[\frac{\nabla(\mu B)}{m}\right] x . \qquad 10$$

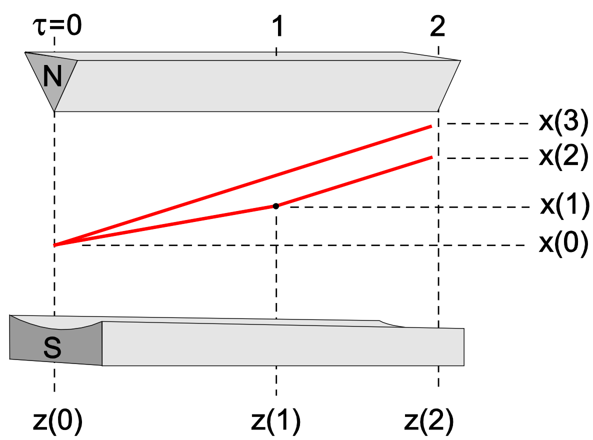


Fig 1. Conceptual diagram of key transition points: $x(0)$ is initial entry to the SG magnetic field, with arbitrary initial orientation. $x(1)$ is the deflection at which the spin is aligned with the local field. $x(2)$ is the deflection when the particle exits the field. $x(3)$ is the maximum possible deflection, which occurs when the particle is aligned with the field upon entry.

Distance $x = x(2) - x(1)$ is the vertical displacement of the aligned particle, so

$$\nabla(\mu B)[x(2) - x(1)] = \frac{mv_x^2(2)}{2} - \frac{mv_x^2(1)}{2} \qquad 11$$

and from energy-exchange we derived $m{v_x}^2(1)/2 = \mu B(1 - \cos\theta)$. When a dipole aligns with the local field there is no longer θ-dependence; the kinetic energy of deflection depends only on the strength of the field, the gradient, the distance traveled, and mv_z.

For incoming aligned particles the outgoing vertical velocity is $v = \nabla(\mu B)\left(L/mv_z\right)$ so for small incoming angles we have $v \approx \nabla(\mu B)\left(L/mv_z\right)$. The vertical distance $x(3)$ traversed by the incoming aligned particle is given by

$$x(3) = \frac{\nabla(\mu B)}{2m}\left(\frac{L}{v_z}\right)^2 \quad \Rightarrow \quad x(2) \approx \frac{\nabla(\mu B)}{2m}\left(\frac{L}{v_z}\right)^2 - x(1)\,. \qquad 12$$

Rewriting equation (11) as

$$[x(2) - x(1)] = \frac{mv^2(2)}{2\nabla(\mu B)} - \frac{\mu B}{\nabla(\mu B)}(1-\cos\theta) \qquad 13$$

and recognizing that as initial angle $\theta \to 0$ we have $x(1) \to 0$ and $x(2) \to x(3)$, we see that, for small angles (compatible with equation 9)

$$x(1) \approx \frac{\mu B}{\nabla(\mu B)}(1-\cos\theta)\,. \qquad 14$$

Thus we derive, for small incoming angles [based on $x(2) \approx x(3) - x(1)$]

$$x(2) \approx \frac{\nabla(\mu B)}{2m}\left(\frac{L}{v_z}\right)^2 - \frac{\mu B}{\nabla(\mu B)}(1-\cos\theta)\,. \qquad 15$$

For any specific Stern-Gerlach experiment the $x(3)$ terms are constants so we can choose them by design such that $x(2) \approx K - K_1(1-\cos\theta)$ where $K_1 = \mu B/\nabla(\mu B)$. Approximating the constant K as the sum of two constants $K = K_0 + K_1$ we obtain

$$x(2) = K_0 + K_1\cos\theta\,, \qquad 16$$

which is a formal Stern-Gerlach deflection for small initial angle of precession. This θ-dependent value of deflection for our classical model generalizes the SG experiment in a relatively geometry independent manner. The actual deflection of course depends upon the geometric scale aspects of the problem in terms of length of travel, strength of field, strength of gradient and initial momentum of the dipole, in addition to the initial angle the dipole makes with the local field.

Normalizing $\mu B/\nabla(\mu B) \to \pm 1$ and choosing K_0 we now simplify the deflection formula:

$$\begin{aligned} x(\theta) &= +1 + \cos\theta \qquad \theta < \pi/2 \\ x(\theta) &= -1 - \cos\theta \qquad \theta > \pi/2 \end{aligned} \qquad 17$$

Theoretical physics is all about writing down models to describe the behaviors of particular systems in the Universe [15]. The above energy-exchange-based formula (17) produces Stern-Gerlach model data. We calculate the deflection x for 1000 random angles and overlay this (red) data on the real data from the iconic postcard that Stern and Gerlach sent to Bohr announcing their discovery of the spin-dependent 'splitting' as shown in fig 2. The asymmetry in the real data is explained later.

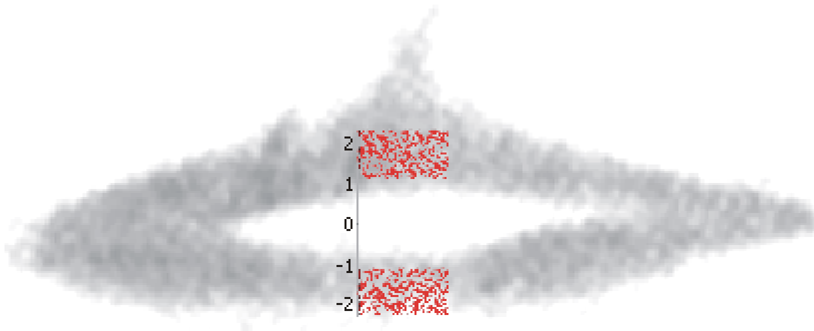


Fig 2. Simulated deflection (red) data points for random initial orientations of the spin are overlaid on the (appropriately scaled) image of the iconic ‘postcard’ data Stern and Gerlach sent to Bohr. The simulated vertical deflection data is spread horizontally merely to show the essentially random distribution. The horizontal spread of simulated data has no other meaning here.

Rate of change of precession

An unanswered question concerning a dipole in an inhomogeneous magnetic field is the rate at which the angle of precession changes. Specifically, how long does it take for the angle of precession to reach zero, i.e., for the dipole to align with the local field. In one sense it doesn't really matter, as long as it's not instantaneous; the rate is a matter of geometry and field strength and strength of gradient. If the alignment were instantaneous, the entire travel through the Stern-Gerlach device would experience the maximum force and the deflection would be to one spot, regardless of the initial angle. This would correspond to Bell’s model; the deflection measurements would yield ± 1. But a finite decay time is angle dependent and the spread of deflection is a function of the initial angle. Absent means of calculating the decay rate of the precession angle to determine the time of alignment, the question *is* answerable experimentally. In fact the original Stern-Gerlach data on the iconic postcard exhibits the spread of deflections expected from energy exchange [16].

In the Stern-Gerlach apparatus the mass of the silver atom has little or nothing to do with the precession, whether viewed as precession angle θ or precession frequency ω. From *the Energy-Exchange theorem* we conclude that precession energy is exchanged with kinetic energy of translation, and the particle is accelerated by force $\vec{F} = \vec{\nabla}(\vec{\mu} \cdot \vec{B})$. Based on an energy conservation approach [$E_{in} = E_{out}$] we derived

$$mv_x^2/2 = \mu B(1 - \cos\theta) \qquad 18$$

where velocity v_x is the velocity in the direction of the gradient (all other components being unchanged) and the angle θ is the initial angle that the dipole makes with the local magnetic field, $\vec{B}$. If we assume spin changes from initial angle θ to alignment

$(\theta = 0)$ instantaneously, this implies that the acceleration is infinite, as it would take no time at all to accelerate from $v_x = 0$ to $v_x = \sqrt{[2\mu B/m](1-\cos\theta)}$. This is physically unrealistic so we conclude that while the rate of precession may be independent of the particle mass, the rate of acceleration is not. Since $\vec{F} = m\vec{a} = \vec{\nabla}(\vec{\mu}\cdot\vec{B})$ and $\vec{a} = \vec{\nabla}(\vec{\mu}\cdot\vec{B})/m$ acceleration is a function of the gradient of the B-field, the angle of precession, and the mass of the particle.

Sec. IV From 'Path Integral' to 'Path'

We focus on *initial and final states of spin*. Quantum 'weirdness' has a long history. Bohr and Heisenberg, based on 'seeing-is-believing', refused to grant 'physical reality' to non-measurable entities. This effectively eliminates 'trajectories', and despite the passage of almost a century, metaphysical concepts underlying quantum mechanics still retain a positivist influence. The goal of our classical model is to retain as much compatibility with quantum mechanics as possible. So to address the aspect that most distinguishes quantum mechanics from classical mechanics, I focus on two *visible* states; *prepared input*, and *measured output*. The Bohr-Heisenberg Copenhagen School rejects the idea of classical-type trajectories, so we avoid assuming continuum $\theta(t)$ and instead consider an *ensemble of various initial states* to derive the differential equations that apply. The physics of a magnetic moment in an inhomogeneous field is not simple. When one attempts to bear in mind and address quantum interpretations of the behavior it becomes decidedly less simple (see Appendix).

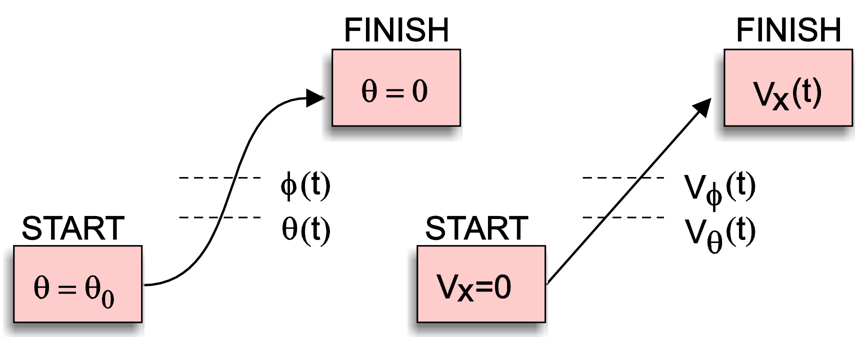


Fig 3. Schematic diagram illustrating that only the *start* and *finish* states are known; the paths are unknown. The Energy-Exchange model is applied to these endpoints to derive the equations of motion describing the actual paths.

The *Energy-Exchange theorem* describes the dynamics of multi-mode energy exchange. The start and finish states are specified as seen in fig 3. We use this to extract the intermediate dynamics. To approach a path-function we first consider two arbitrary angles ϕ and θ, and calculate the distance between their respective endpoints by subtracting the θ-dynamics from the ϕ-dynamics. We next bring the points together, only an infinitesimal distance apart, such that $\phi = \theta - d\theta$. We then use this relation to refine the dynamics along the spin path. Both $\theta(t)$ and $\phi(t)$ are variables that define starting points on the path from $\phi(0)$ and $\theta(0)$ to $\phi(t) = 0$ and $\theta(t) = 0$. We apply the *Energy-Exchange* approach which assumes initial angle θ and final alignment $\theta = 0$. We would like to generalize the treatment to handle variable θ, ideally as a function of time. We begin by analyzing energy exchange for two different angles θ and ϕ:

$$\frac{1}{2} m v_\phi^2 = \mu B(1 - \cos\phi) \tag{19}$$

$$\frac{1}{2} m v_\theta^2 = \mu B(1 - \cos\theta) \tag{20}$$

The difference is

$$\frac{1}{2} m [v_\phi^2 - v_\theta^2] = -\mu B\,(\cos\phi - \cos\theta)\,. \tag{21}$$

Assume that precession angle ϕ differs infinitesimally from θ:

$$\phi = \theta - d\theta \tag{22}$$

so the corresponding equation becomes

$$\frac{1}{2} m [v_\phi^2 - v_\theta^2] = 2\mu B \sin\left(\frac{2\theta}{2}\right)\sin\left(\frac{-d\theta}{2}\right) \tag{23}$$

But $\sin\left(\frac{-d\theta}{2}\right) = -\frac{d\theta}{2}$, therefore $\frac{1}{2} m [v_\phi^2 - v_\theta^2] = -\mu B \sin\theta\, d\theta$ for $d\theta > 0$. That is, for increasing precession angle the translation energy decreases. The physical process is instead such that when the precession angle decreases ($d\theta < 0$) the particle velocity increases. So we can formulate v_ϕ for positive $d\theta$ based on the fact that $dv/d\theta$ is negative (where v is velocity in vertical direction):

$$v_\phi = v(\theta + d\theta) = v(\theta) + \frac{dv}{d\theta} d\theta \quad \Rightarrow \quad v_\phi{}^2 = [v(\theta) + \frac{dv}{d\theta} d\theta]^2\,, \tag{24}$$

and

$$[v_\phi^2 - v_\theta^2] = +2v(\theta)\frac{dv}{d\theta} d\theta \quad \Rightarrow \quad \frac{1}{2} m [2v(\theta)\frac{dv}{d\theta} d\theta] = -\mu B \sin\theta\, d\theta \tag{25}$$

so

$$mv\frac{dv}{d\theta} = -\mu B \sin\theta \quad \Rightarrow \quad \underbrace{mv\,dv}_{FINI} = \underbrace{-\mu B \sin\theta\, d\theta}_{INIT}\,. \tag{26}$$

This is the key equation derived from the analysis of the two angles. We now integrate from arbitrary initial angle θ to final alignment $(\theta \to 0)$:

$$\int_{v(\theta)=0}^{v} mv\,dv = -\mu B \int_{\theta}^{0} \sin\theta\,d\theta \qquad 27$$

We recover the energy exchange result we started with, which is a good sign [17]:

$$mv^2/2\Big|_0^v = -\mu B(-\cos\theta)\Big|_\theta^0 \quad \Rightarrow \quad mv^2/2 = \mu B\,(1-\cos\theta)\,. \qquad 28$$

Thus, beginning with the relation of *final velocity* to *initial angle*, we used an ensemble of all angles to treat two angles that differed infinitesimally. The resultant differential equation in terms of variable v and variable θ yield the correct endpoint results for initial θ and final v. We are thus encouraged to investigate further using standard treatment of variables, including the time dependence of the precession angle. We first note the relevant dependencies:

$$v(x,\theta,t), \quad B(x), \quad x(t), \quad \theta(t).$$

Based on these dependencies we take the derivatives of $mv^2/2 = \mu B(1-\cos\theta)$. The first derivative with respect to θ yields

$$mv\frac{dv}{d\theta} = \mu B\sin\theta\,. \qquad 29$$

This is the correct mathematical result, but it conflicts with the relation we just derived and checked. The problem is that the default calculus interpretation of $dv/d\theta$ is positive, while physically we know that $dv/d\theta < 0$. Therefore we *make the sign explicit* and recover our key equation in the standard interpretation:

$$mv\frac{dv}{d\theta} = -\mu B\sin\theta\,. \qquad 30$$

We next take the derivative of translational energy with respect to deflection x:

$$mv\frac{dv}{dx} = \mu\frac{\partial B}{\partial x}(1-\cos\theta) \qquad 31$$

The velocity dv is positive with respect to displacement dx so this derivative appears to be correct. Finally, we take the time derivative of the energy exchange equation to obtain:

$$mv\frac{dv}{dt} = \mu\frac{\partial B}{\partial x}\frac{dx}{dt}(1-\cos\theta) + \mu B\sin\theta\frac{d\theta}{dt}\,. \qquad 32$$

Sec. V Precession Decay Rate

From eqn (32) we can solve for the rate of change of the precession angle, $d\theta/dt$. To proceed we note that $dx/dt = v$ and $dv/dt = a = \frac{\mu}{m}\frac{\partial B}{\partial x}$, hence

$$\left(\frac{\mu}{m}\right)mv\frac{\partial B}{\partial x} = \mu v\frac{\partial B}{\partial x}(1-\cos\theta) + \mu B\sin\theta\frac{d\theta}{dt} \qquad 33$$

or

$$\mu v\frac{\partial B}{\partial x} - \mu v\frac{\partial B}{\partial x} = -\mu v\frac{\partial B}{\partial x}\cos\theta + \mu B\sin\theta\frac{d\theta}{dt} \quad = 0 \qquad 34$$

Thus

$$\frac{d\theta}{dt} = \left(\frac{v}{B}\right)\left(\frac{\partial B}{\partial x}\right)\left(\frac{\cos\theta}{\sin\theta}\right). \qquad 35$$

Once again we are faced with a default calculus positive interpretation of a physical phenomenon that is negative: $d\theta/dt < 0$. We again make the sign explicit as follows:

$$\frac{d\theta}{dt} = -\left(\frac{v}{B}\right)\left(\frac{\partial B}{\partial x}\right)\left(\frac{\cos\theta}{\sin\theta}\right). \qquad 36$$

This is the precession decay rate. To simplify it we can derive the maximum value of the velocity. We ask for the angle corresponding to maximum energy by taking the derivative of the energy and setting it equal to zero:

$$\frac{d}{d\theta}\left[\frac{1}{2}mv^2\right] = \frac{d}{d\theta}\left[\mu B(1-\cos\theta)\right] \equiv 0 \quad \rightarrow \quad \left[\mu B\sin\theta\right] \equiv 0 \quad \Rightarrow \quad \theta = 0. \qquad 37$$

That is, maximum energy of translation occurs when $\theta = 0$. For fixed particle mass this corresponds to maximum velocity v, thus the maximum velocities occur for small angles θ. Substituting the small angle relation $\cos\theta \approx 1-\theta^2/2\cdots$ into energy equation

$$mv^2/2 = \mu B(1-\cos\theta) \quad \xrightarrow{\theta\to 0} \quad \underbrace{mv^2}_{FINI} = \underbrace{\mu B\theta^2}_{INIT} \qquad 38$$

and setting $k = \sqrt{\mu/m}$ we obtain

$$v \approx k\sqrt{B}\,\theta \quad \sim \quad k\sqrt{B}\sin\theta. \qquad 39$$

Substituting this approximate value into the derivation of $d\theta/dt$ we obtain

$$\frac{d\theta}{dt} \approx -\frac{k}{\sqrt{B}}\frac{\partial B}{\partial x}\cos\theta. \qquad 40$$

For small angle θ we use $\cos\theta \sim 1$ to obtain

$$\frac{d\theta}{dt} \approx -\frac{k}{\sqrt{B}}\frac{\partial B}{\partial x} \qquad \Rightarrow \qquad \frac{d\theta}{dt} = -2k\frac{\partial\sqrt{B}}{\partial x}. \qquad 41$$

From this small angle approximation we observe that the *decay rate increases as the gradient increases*; the rate at which a particle will align with the local field increases as the particle moves into the stronger field region.

We began by asking how quickly the dipole aligns with the local field in the Stern-Gerlach experiment, noting that, if alignment were instantaneous, then all particles would immediately experience the maximum force and would be deflected to the maximum position, effectively agreeing with Bell's assigning all measurements to ± 1. This immediate alignment would in some way correspond to the quantum mechanical 'collapse of the wave function'. Instead, we find that the decay of precession angle θ is given by equation (41).

This decay dynamic yields a *new physical result* so it is desirable to check it. Let us re-examine our key difference equation:

$$\frac{1}{2}m[v_\phi^2 - v_\theta^2] = -\mu B \sin\theta \, d\theta. \qquad 42$$

The first term clearly resembles the distance formula $v_f^2 = v_i^2 + 2ax$ where v_f is the final velocity, v_i is the initial velocity, a is the acceleration and x is the distance traveled, so we can replace $v_\phi^2 - v_\theta^2$ by $2ax$. When the velocity corresponds to the differential $\phi - \theta = -d\theta$, the distance x is the differential dx traveled by the particle as it varies from precession angle θ to angle $\theta - d\theta$.

$$\frac{1}{2}m[v_f^2 - v_i^2] = \frac{1}{2}m[2a\,x] \Rightarrow ma\,dx = F\,dx \qquad (= work) \qquad 43$$

But the force $\vec{F} = m\vec{a} = \vec{\nabla}(\vec{\mu}\cdot\vec{B})$ implies that $\vec{a} = \vec{\nabla}(\vec{\mu}\cdot\vec{B})/m$, hence

$$m\vec{a}\,dx = \vec{\nabla}(\vec{\mu}\cdot\vec{B})\,dx \sim \mu\cos\theta\frac{\partial B}{\partial x}dx \qquad 44$$

and so

$$\mu\cos\theta\frac{\partial B}{\partial x}dx = -\mu B\sin\theta\,d\theta \qquad 45$$

or

$$\frac{d\theta}{dx} = -\left(\frac{1}{B}\right)\frac{\partial B}{\partial x}\left(\frac{\cos\theta}{\sin\theta}\right). \qquad \Rightarrow \qquad \tan\theta\,d\theta = -\left(\frac{1}{B}\right)\frac{\partial B}{\partial x}dx \qquad 46$$

The negative sign correctly implies that θ decreases as deflection x increases. But the derivative $d\theta/dx$ is an artifact, since θ is not a direct function of x, nor is x assumed to be a function of θ. Rather, both θ and x are functions of time. We reformulate $d\theta/dx$ as $\frac{d\theta/dt}{dx/dt}$ and so we can multiply $d\theta/dx$ by velocity v to obtain $d\theta/dt$, i.e.

$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt} = v\frac{d\theta}{dx} \qquad 47$$

where, based on equations (36) and (46), we see that

$$\frac{d\theta}{dx} = -\left(\frac{1}{B}\right)\frac{\partial B}{\partial x}\left(\frac{\cos\theta}{\sin\theta}\right) \quad \rightarrow \quad \frac{d\theta}{dt} = -\left(\frac{v}{B}\right)\left(\frac{\partial B}{\partial x}\right)\left(\frac{\cos\theta}{\sin\theta}\right) \equiv v\frac{d\theta}{dx}. \qquad \text{QED} \qquad 48$$

So *decay of precession is not instantaneous*, and thus does not correspond to "*collapse of the wave function*". Nor does it lead to a single point of deflection that could legitimately be characterized as a +1 measurement (-1 for $\theta > \pi/2$).

Based on the *Energy-Exchange analysis* we have calculated the particle velocity (in the deflection direction) and vertical displacement at the point where the dipole becomes aligned with the local field. We compute the horizontal position at which this alignment occurs via fixed velocity v_z: we use $z = v_z\tau$ where τ is the time at alignment.

$$x(1) \cong \frac{\mu B}{\mu\nabla B}(1-\cos\theta), \quad v_x^2 = \frac{2\mu B}{m}(1-\cos\theta), \quad z = v_z\tau, \quad \frac{d\theta}{dt} = -\frac{k}{\sqrt{B}}\frac{\partial B}{\partial x}. \qquad 49$$

As both displacement x and decay $\dot{\theta}$ depend on the gradient, we must make assumptions about the field gradient. A common assumption is that used by Griffiths [1] p.193

$$\vec{B}(x,y,z) = (B_0 + \alpha x)\,\hat{i} - \alpha y\,\hat{j}, \qquad 50$$

which is chosen to satisfy Maxwell's $\vec{\nabla}\cdot\vec{B} = 0$. As we are interested only in the vertical deflection, we assume the particle is initially aligned such that no *y* component force is significant. This allows the useful approximation $\partial B/\partial x = \alpha$. Using this we obtain

$$x(1) \cong \frac{B}{\alpha}(1-\cos\theta), \qquad v_x = \sqrt{\frac{2\mu B}{m}(1-\cos\theta)} \approx \sqrt{\frac{\mu B}{m}\theta^2} = \sqrt{\frac{\mu B}{m}}\theta = k\sqrt{B}\theta, \qquad 51$$

$$\frac{d\theta}{dt} = -\frac{k}{\sqrt{B}}\frac{\partial B}{\partial x} = -\frac{k\alpha}{\sqrt{B}} \quad \Rightarrow \quad t = \left(\frac{\sqrt{B}}{k\alpha}\right)\theta \qquad 52$$

Interestingly, each of the expressions for $(x(1), v_x, t)$ are dependent on θ. Thus the $x(1), z, v_x, v_z$, and t parameters are approximately known when the dipole aligns with the local field, and these parameters allow us to compute the deflection as a function of time using simple distance formula $x(t) = x(1) + v_x t + a t^2/2$, where the acceleration $a = F/m = \mu\nabla B/m = k^2\alpha$ since the force on aligned particles is independent of initial angle θ. From this analysis θ-dependent paths through Stern-Gerlach apparatus for small angles θ [deg] are seen in fig 4; dotted curves begin at points $(x(1), z(1))$. Shaded areas represent regions in which spin is aligning with the local field, while dotted curves represent maximum acceleration of aligned spins. This scale is chosen to accentuate the differences. Actual scale factors are field and geometry dependent.

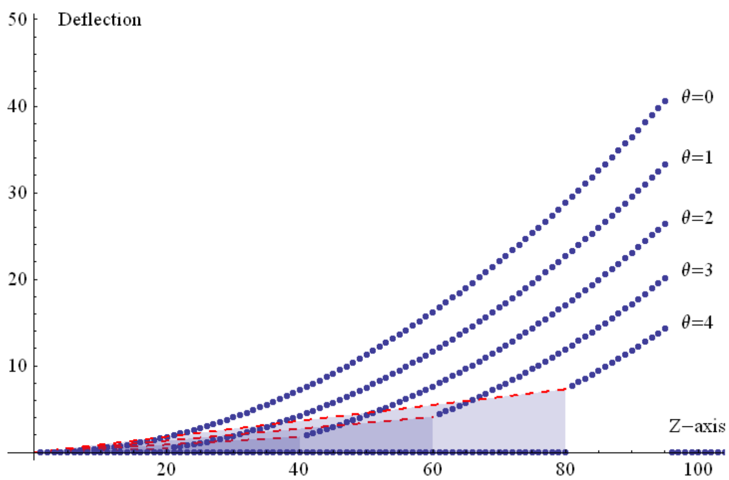


Fig 4. The θ-dependence for small angles is shown. Shaded areas represent the alignment process during which the initial spin enters the field with angle θ and proceeds to align with the local field. After alignment the gradient force accelerates all particles equally. The scales have been chosen to enhance the different deflections. Actual scales depend upon field strength, field gradient, length of travel, and geometry.

Otis Lamont Frost [29] solved equation (46) for displacement x by integrating from initial condition θ_i to time t:

$$dx = -\frac{B}{\partial B/\partial x}\tan\theta\, d\theta \quad \Rightarrow \quad x(t) = \frac{B}{\partial B/\partial x}\ln\left(\frac{\cos\theta(t)}{\cos\theta_i}\right) \qquad 53$$

At alignment $\theta(t) = 0$ so the exact expression for deflection at alignment is

$$x_f = -\frac{B}{\partial B/\partial x}\ln(\cos\theta_i)\,. \qquad 54$$

Based on $\ln(x) \approx x - 1$ this reduces to small angle approximation equation (49a):

$$x_f \approx \frac{B}{\partial B/\partial x}(1-\cos\theta_i)$$

In this context the deflection for <u>all</u> angles yields the 'lips' pattern; the 'small approximation' formula is good to about 45 degrees, which is the angle at which most Bell-tests are performed (albeit on photons).

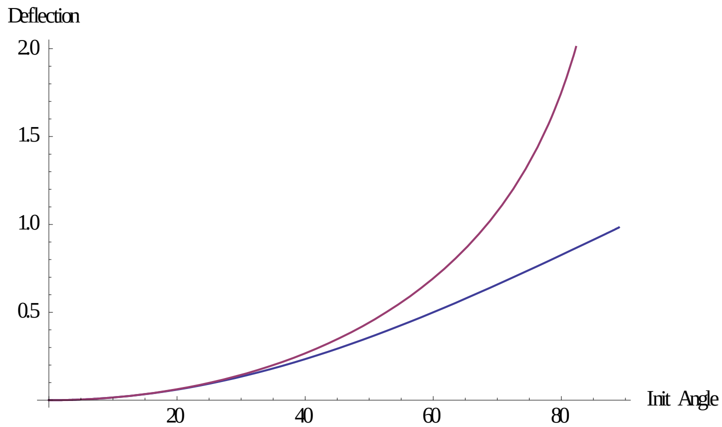


Fig. 5 The exact solution to eqn (46) yields eqn (54). This comparison to the small angle approximation (blue) implies that the small angle approximation is reasonably accurate up to almost 45 degrees.

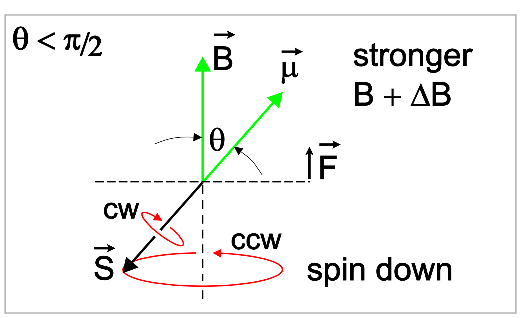
Sec. VI Dynamic Spin Stability

To address Bell's statement about "*compass needles pointing in the wrong direction*" being "*not dynamically sound*" requires making use of Deissler's observation about spin and precessional rotation. It additionally requires an asymmetry that is foreign to Bell's analysis. The asymmetry is best understood by comparison with the assumptions of first-order energy exchange, namely

Symmetric	Asymmetric	
$E_{in} = E_{out}$	$E_{in} = E_{out}$	55
$B_{in} = B_{out}$	$B_{out} = B_{in} + \Delta B$	

The first-order approximation energy-exchange model with instantaneous decay of precession leads to Bell-like measurement spectrum of ± 1. The finite decay model leads to a symmetric spread spectrum. A second-order gradient-based approximation yields an asymmetric energy-exchange model. To develop the asymmetric model we explicitly consider 'spin-up' and 'spin-down' cases.

Spin down $\theta \leq \pi/2$

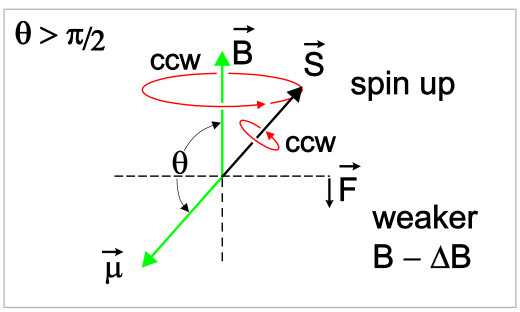


Because the dipole moment $\vec{\mu}$ is the product of electronic charge and spin, the negative electron implies $\vec{\mu} = -\vec{s}$ as shown and spin-down precession (CCW) is opposite the rotation of the spin (CW). As the force on the dipole accelerates the particle to the region of stronger B-field, $B \to B + \Delta B$, the (CCW) precession frequency increases. This opposes the intrinsic CW spin and thus decreases the kinetic rotational energy:

$$\begin{Bmatrix} cw \\ spin \end{Bmatrix} + \begin{Bmatrix} ccw + \Delta ccw \\ precession \end{Bmatrix} = \begin{Bmatrix} rot.\,energy \\ \Downarrow \end{Bmatrix}$$

Thus as $\vec{B}$ increases, the $-\vec{\mu} \cdot \vec{B}$ energy becomes more negative. If energy is conserved, the translational kinetic energy $mv_x^2/2$ becomes more positive as the particle is accelerated upward. This agrees with the decrease in rotational kinetic energy as shown. We next consider the opposite spin.

Spin up $\theta \geq \pi/2$



A particle with spin up is accelerated into a region of weaker local magnetic field so the precession frequency (energy) decreases. Since the (CCW) spin and (CCW) precession are in the same direction the net rotational energy decreases:

$$\begin{Bmatrix} ccw \\ spin \end{Bmatrix} + \begin{Bmatrix} ccw - \Delta ccw \\ precession \end{Bmatrix} = \begin{Bmatrix} rot.\,energy \\ \Downarrow \end{Bmatrix}$$

Thus as $\vec{B}$ decreases the $\vec{\mu} \cdot \vec{B}$ energy becomes smaller (less positive) and the translational energy $mv_x^2/2$ grows as the rotational energy decreases.

Hence for all angles of the initial dipole with respect to the magnetic field

$$\underset{rotational}{\begin{Bmatrix} I\omega^2/2 \\ \Downarrow \end{Bmatrix}} + \underset{translational}{\begin{Bmatrix} mv_x^2/2 \\ \Uparrow \end{Bmatrix}} = \begin{matrix} energy \\ balance \end{matrix}$$

With this more detailed analysis of the spin dynamics we now reanalyze the energy balance equation $E_{in} = E_{out}$. Having concluded that there is no change in the z-component of velocity, we know that it cancels and we also assume that the incoming x-velocity is zero, thus we retain our energy exchange equation

$$-\vec{\mu}\cdot\vec{B}_{in} = mv_x^2/2 - |\vec{\mu}||\vec{B}_{out}| \qquad 56$$

We again consider two cases $\theta \le \pi/2$ and $\theta \ge \pi/2$.

Our initial energy exchange analysis explicitly conserved energy, $E_{in} = E_{out}$, but simplistically assumed $B_{in} = B_{out}$. A better approximation assumes $B_{out} = B_{in} \pm \Delta B$ in order to take a necessary gradient into account. We now explore the spin dynamics of energy exchange in an explicit gradient formulation. It is again necessary to treat the spin up and spin down cases separately, as the $\cos\theta$ term yields a sign change between these two cases.

For $\theta < \pi/2$ the dipole experiences a positive force toward a region where the field is stronger. The initial angle between the spin and the local field is θ and the final aligned state has zero angle. We begin by assuming local conservation of energy:

$$E_{in} \quad = \quad E_{out} \qquad 57$$

$$-\vec{\mu}\cdot\vec{B}_{in} = mv_x^2/2 \; - |\mu||B_{out}| \qquad 58$$

Let $B_{out} = B_{in} + \Delta B$ where $\Delta B = \int_0^x \frac{dB}{dx}dx$ 59

$$mv_x^2/2 \; = |\mu||B_{out}| - \vec{\mu}\cdot\vec{B}_{in} \quad = \quad |\mu|(B_{in} + \Delta B) - \vec{\mu}\cdot\vec{B}_{in} \qquad 60$$

$$mv_x^2/2 \; = |\mu||B_{in}|(1-\cos\theta) + \mu\Delta B\,. \qquad 61$$

All terms are positive and thus the kinetic energy is positive as required. The $\mu\Delta B$ additive energy represents greater kinetic energy (and correspondingly greater deflection) than the simpler symmetric approximation.

For $\theta > \pi/2$ the dipole experiences a force toward a weaker region of the field. Here the cosine is negative, so $\vec{\mu}\cdot\vec{B}$ is negative and energy $-\vec{\mu}\cdot\vec{B}$ is therefore positive.

$$E_{in} \quad = \quad E_{out} \qquad 62$$

$$-\vec{\mu}\cdot\vec{B}_{in} = mv_x^2/2 \; - |\mu||B_{out}|(\cos\pi) \qquad 63$$

Taking signs into account, this becomes:

$$|\vec{\mu}||\vec{B}_{in}||\cos\theta| = mv_x^2/2 + |\mu||B_{out}| \qquad 64$$

$$mv_x^2/2 = |\vec{\mu}||\vec{B}_{in}||\cos\theta| - |\mu||B_{out}| \qquad 65$$

But, due to the gradient, $B_{out} < B_{in}$ so $B_{out} = B_{in} - \Delta B$

$$mv_x^2/2 = |\vec{\mu}||\vec{B}_{in}|(|\cos\theta| - 1) + \mu\Delta B \quad \Rightarrow \quad \Delta B > |B(|\cos\theta| - 1)| \qquad 66$$

The term $(|\cos\theta| - 1)$ is *always* negative, so the requirement of positive kinetic energy of translation $mv_x^2/2$ demands the $\mu\Delta B$ term be sufficiently positive to overcome the negative term. Thus equation (64) implies a threshold gradient, *below which the Stern-Gerlach apparatus will not work*. Equations (64) and (66) show the (integral of) the gradient to be a function of θ and of translational velocity v:

$$\Delta B = B(1 - |\cos\theta|) + \frac{1}{2}\left(\frac{v^2}{k^2}\right) \quad \Rightarrow \quad \frac{mv^2}{2} > 0 \qquad 67$$

The above analysis enables *a new physical conclusion*, which is the existence of a threshold of inhomogeneity required for Stern-Gerlach apparatus. Below this threshold the beam will not split, in accord with Bell's statement that "the compass needles" would be pointing in the wrong direction and hence "not dynamically sound".

Sec. VII The Asymmetric Approximation

A finite decay rate implies a θ-dependent spread of SG deflections, in contrast to the single data point expected if alignment were instantaneous. The asymmetric treatment displays the second-order effects of the stronger and weaker regions of gradient. The combination of these classical effects is shown in fig 6 overlapping the real Stern-Gerlach data from the iconic postcard on which we have overlaid the gradient-producing magnetic field geometry. So far we have considered only $\partial B/\partial x$, i.e., vertical deflection, but this is incompatible with Maxwell's $\vec{\nabla} \cdot \vec{B} = 0$ [1][12][25]. Inclusion of a y-axis term provides the left/right displacements shown in the data, but adds no new physics insight to the model.

A typical asymmetric Stern-Gerlach *x*-deflection for random initial angle θ is next calculated by rewriting equation (17) to accommodate the asymmetry via β and $\beta/3$:

$$x = +1 + \beta\cos\theta \qquad \theta < \pi/2 \qquad 68$$

$$x = -1 - (\beta/3)\cos\theta \qquad \theta > \pi/2 \qquad 69$$

The scaling value $\beta = 4$ is chosen to yield a best match to the iconic postcard data. The actual deflection of course depends on the strength of the field, the strength of the gradient, the initial angle, and velocity, the length of the SG-magnet and the distance from the magnet to the detection screen. We effectively normalize the deflection by choosing the deflection due to the magnet-to-screen travel to be +1 or -1. In similar fashion, we assume the region of high-strength gradient to be a multiple of the low strength gradient. The values can be approximated via appropriate strengths and geometries. If we vary the spin angle θ randomly and plot the vertical deflections based on equations (66) and (67) we obtain the asymmetric distribution. The (red) calculated random data points are scaled and overlaid on the SG iconic data as seen in fig 6. The match between calculated data and experimental data is extremely good.

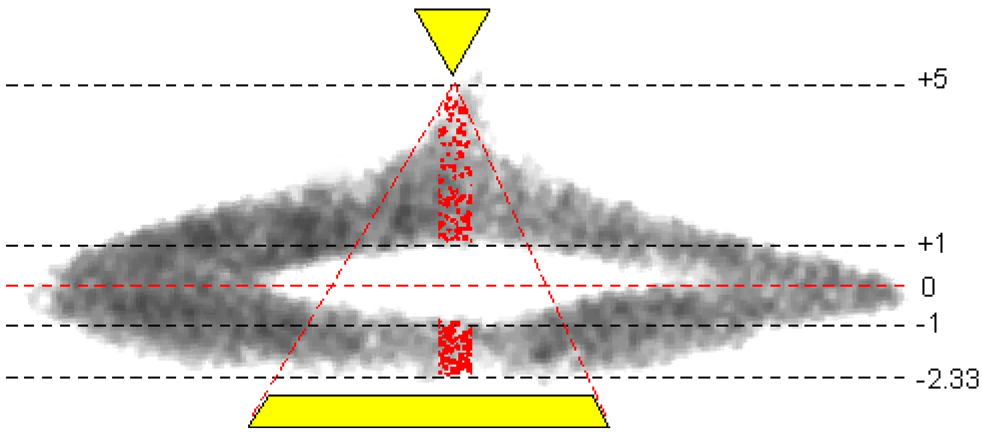


Fig 6. The θ-based model of spin deflection in the Stern-Gerlach experiment is driven with random spin vectors and trajectories are calculated. The red dots representing individual particles are overlaid on the gray SG data from the iconic postcard. The scale has been chosen to facilitate the overlay. The Stern-Gerlach magnets are diagrammed in yellow. Horizontal spreading of the red data points is for illustrative clarification and is not physically meaningful.

Sec. VIII Another classical treatment of Stern-Gerlach: the Edge-Effect

In contrast to the above analysis, Franca [27] does not distinguish between *free energy exchange* and the *work performed by the magnet*, and assumes that polarization of the atom is modified "*in a very restricted region at the entrance of the magnetic pole pieces*." In this case acceleration by the inhomogeneous field on the atom during the transit of the field could be the same for all atoms, leading to a 'point' on the target screen, corresponding to *collapse of the wave function* or *immediate alignment* if *all* initial orientations effectively experience exactly the same acceleration and are thus directed to a point on the detector rather than being smeared as the iconic pattern shown by Stern-Gerlach. *Edge-effect* models are relatively insensitive to the nature of the bulk magnetic field; an incoming particle experiences some force in the velocity direction when it enters the field, independent of whether the field is homogeneous or inhomogeneous, at least to first-order. Franca (p.1182) states:

> *"We shall show that this* ["*edge-effect*"] *property is responsible for the Stern-Gerlach phenomenon."*

Franca's analysis is *classical*, not *quantum*, yet uses *probability distributions* to explain Stern-Gerlach. The gradient on entry to the field is dominant. If the *x*-axis is the direction of the field and the *z*-axis is the initial velocity, his concern is with the *z*-axis gradient at the edge of the field, with a consequent change in *z*-velocity. Stern-Gerlach ignores edge-effects so z-velocity remains constant. Franca establishes a $\dot{\theta}$ equation based on an *energy-exchange-like* approach to energy conservation in the z-direction as the particle enters the field on the z-axis, expressing the initially random θ-dependence as polarization:

$$P_i(\theta_i) = \frac{\sin\theta}{2} \qquad 70$$

Immediately following this equation for *polarization* $P_i(\theta_i)$ he switches to a discussion of *probability* $P(\theta,t)$ based on probability conservation over time

$$\frac{dP(\theta,t)}{dt} = 0 \quad \Rightarrow \quad \frac{\partial P(\theta,t)}{\partial t} + \frac{d\theta}{dt}\frac{\partial P(\theta,t)}{\partial \theta} = 0 \qquad 71$$

then he plugs in his formula (eqn 14) for $d\theta/dt$ which can be compared to our (48):

$$\frac{d\theta}{dt} = -\gamma(v_x)\cot(\theta) \qquad 72$$

where γ is a function of the *z*-component of the field B_z and its change in the *x*-direction $\partial B_z/\partial x$ as the particle enters. He obtains an order of magnitude estimate of the factor γ based on Rabi's work — Stern and Gerlach did not measure this value.

Franca claims that "*The change in the orientation of the magnetic dipole, with the space variation of the magnetic field, at the entrance of the magnet*" is the important aspect of Stern-Gerlach, though he only credits it for producing a *partial polarization* of the beam. His physical basis, the *gradient* (of the edge field), is present in the interior of the Stern-Gerlach field, so it is not at all clear that edge effects are determinative.

Franca's orientation probability distribution $P_i(\theta_i)$ and his probability $P(\theta,t)$ are both "*statistically independent functions by construction*" since $P_i(\theta_i)$ is the initially random orientation from the oven, while $P(\theta,t)$ is the probability of orientation θ based on θ changing due to the edge-effect. It characterizes his *modified* distribution of particles as they traverse the Stern-Gerlach field. This biases the randomly generated initial probability by a θ-dependent factor which depends on the edge effect. Beyond this point the analysis of the trajectory through the device proceeds as usual, though now complicated by the increment of velocity in the *y*-direction, as well as the *no-longer-random* distribution of θ_i. The result is the skewed distribution shown in his figure 5, which is supposed to account for the splitting of the beam. I have shown above that the *beam splits even when the edge effect is ignored.*

Of significance from an experimental perspective; Franca applies *two* probabilities to derive a resultant probability distribution that can be said to *bear some resemblance to the actual data.* The 'double maxima' distribution is obtained by generally reasonable assumptions, but the accuracy of the calculations and the accuracy of measured distributions are insufficient to prove more than 'resemblance' to each other.

Our proposed experiment does not depend on statistics, but is based on *measurement of individual particle trajectories* predicted by the energy-exchange theory including both *free energy* exchange and the *work* imposed on the aligned particle by the field.

Instead of analyzing a *random initial distribution* skewed by an edge effect, our experiment *prepares specific orientations*, θ_i, and predicts specific deflections. These are based on individual measurements, in which the angle θ_i is varied in controlled fashion, and there is *no need for the probability distribution of the input particles*. Nor, if the experiment confirms expectations, is there a need to characterize the resultant deflection probabilistically, as the classical model is deterministic.

Sec. IX Summary and conclusion

The Stern-Gerlach experiment was the basis of the concept of intrinsic spin-½. [30]

> *"A spin-½ system represents the most fundamental quantum mechanical object* [*but*] *experiments with well-controlled and adjustable environments are scarce."*

If the iconic postcard data represents actual physics occurring in an inhomogeneous magnetic field, our model accurately depicts the spin dynamics expected from the SG experiment. From Bell's own words and the historical sequence of analysis, most of which occurred long after Bell's theorem, it is clear that none of the key aspects were known to Bell. Our *energy-exchange model* based on modern insights adds *work* to Bell's model that is *not* accounted for in standard quantum treatments.

A major conceptual aspect of quantum spin is based on Goudsmit's 1925 statement

> *"The projection of spin on any axis is* ± 1*."*

He probably arrived at his statement from Bohr orbits, with quantized energy n and angular momentum l, whose projection of angular momentum on the z-axis is $\pm m\hbar$. But this is an essential example of quantum weirdness, since we cannot geometrically *picture*, hence not *imagine*, a physical spin or quantized entity whose projection on *any* axis is ± 1. This has led to statements [31] as: "*...when we measure a particle's component of spin in a* [any] *particular direction in a Stern-Gerlach experiment, it is the general belief that we are not measuring a pre-existing property*."

Based on the results of the Stern-Gerlach experiment, in which the *z*-axis can point in essentially *any* direction, Goudsmit concluded that the projection of spin on any axis is quantized, and for a spin-½ particle, is $\pm\hbar/2$, as shown in fig 7. This is formalized as $\hat{\sigma}\,|\pm\rangle = \pm|\pm\rangle$ which is effectively independent of spin 'direction' in three dimensions and is the basis of the 'qubit' or two-state conception of spin.

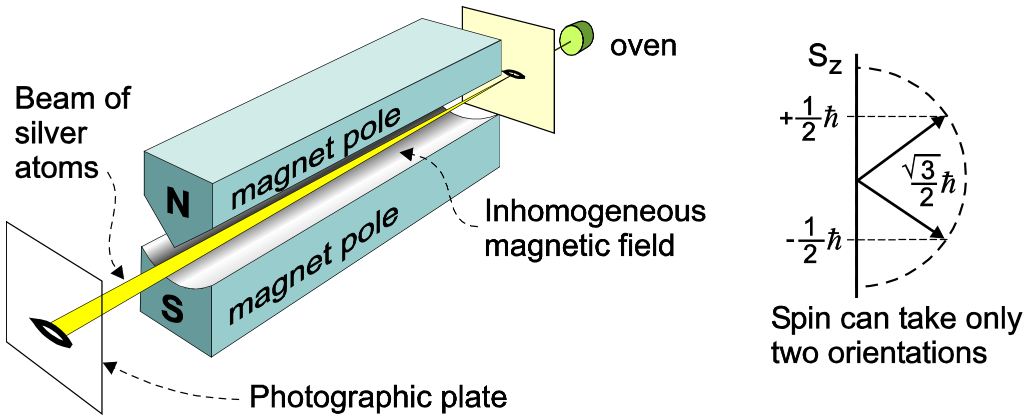


Fig 7 Quantized Bohr orbits were well known to Goudsmit and Uhlenbeck in 1925 (as Heisenberg and Schrödinger were inventing quantum mechanics) and are characterized by the projection $\pm m\hbar$ of angular momentum on the *z*-axis.

This two-state conception is appropriate for the Stern-Gerlach interpretation of beam-splitting, the Pauli *Exclusion Principle* for two electrons in lowest orbit, the 'spin-flip' of precession by photon, and magnetic domains in which particles align or anti-align. The unqualified success of a two-state spin interpretation, and the quantum formalism developed for it has been sufficient to keep the Goudsmit *'any axis principle*' alive for almost a century. But, with the exception of photon-spin flip, all these examples involve *multiple* spins interacting with the field or with each other. And spin flip, based on a spin precessing in a constant local field, is compatible with the examples.

How does this change if the local field is *nonuniform* as analyzed herein? Our theory implies that the classical 3D spin *is* appropriate and that 3D geometry of the initial spin direction and the local field direction *determines* the path through the field. It further implies that the *Goudsmit principle* is false. Only in a *constant* field is the projection on the local axis $\pm\hbar/2$. In a nonuniform field the projection of the spin varies while the spin evolves from initial 3D orientation to alignment. This classical model differs significantly from the *qubit* model. In principle, this distinction can be experimentally tested, as we discuss next.

Ghirardi and Romano [32] ask:

> *"Could a theory, which is conceived as a completion of quantum mechanics, be experimentally distinguishable from it? By completion we mean that the theory should be consistent with quantum mechanics; that is it should fully reproduce all the quantum outcomes in a suitable regime, but it could provide a more refined description of the microscopic reality."*

In their letter they prove that "*ontological models of quantum theory which are compatible with it but, possibly distinguishable from it, are possible.*"

In view of the significance of the Stern-Gerlach experiment for fundamental quantum mechanics and the fact that the classical spin dynamical model developed here differs from the historical version of SG spin, *the 1922 Stern-Gerlach experiment should be performed with 2016 technology and techniques*. The goal: measure the θ-dependence of *single atoms* as opposed to the 'beam-splitting' of the original experiment.

An accurate atomic measurement confirming the spin model described herein would falsify certain current quantum beliefs as well as a well-known *two-slit spin analog* that has only the authority of a gedanken experiment, but is nevertheless presented as the *basis* of several key texts on quantum mechanics. [18][19][20]

In short, there may be a reason that, almost a century after quantum mechanics was developed, it is still said that no one understands QM. A recent "loophole free" test of Bell's theorem [26] is based on assumptions of spin dynamics that conflict with our classical model. For these and other reasons it is suggested that single-atom-SG experiments could either confirm or condemn several currently widely held beliefs about quantum mechanics. A schematic of such an experiment is shown in fig 8.

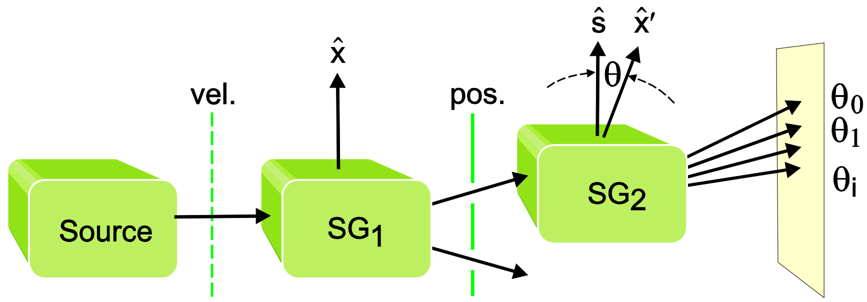


Fig 8. The proposed modern Stern-Gerlach experiment is diagrammed. The source at left is an oven that produces atoms with one electron spin. The velocity filter uses time-of-flight to select a tightly defined velocity. The SG1 apparatus prepares atoms with spin aligned with $\vec{x}$, the local SG1 axis. A position filter is applied to the output of SG1 and those atoms passing this filter are input to SG2, which is oriented at angle θ with respect to SG1. For a given angle θ the deflection is predicted as shown. Successful operation of the experiment depends upon a 'single atom' position detector which will yield the deflection on an atom-by-atom basis.

A Stern-Gerlach measurement of classical spin *directly* measures deflection, that is, the scattering of the dipole by an inhomogeneous magnetic field. The model predicts each measurement deterministically, based on the angle that the local spin makes with the SG field alignment. A quantum mechanical model makes no individual predictions but instead predicts a probabilistic *average* or *expectation value*, $\langle x \rangle$.

The term 'expectation value' is well defined in QM. Nevertheless, Jaynes argued that the probabilistic reasoning in Bell's theorem does *not* follow rules of probability theory [21]. And Wang has analyzed Bell's definition of expectation value in detail and found that it differs from the QM definition of expectation value [22]. Although this appears to be obviously true, all mathematical arguments against Bell's theorem tend to suffer benign neglect. But a key result is that *our model predicts measurements of individual experiments* that are independent of the differences in definition of expectation value.

The *Quantum Credo* is the belief that 'reality' is quantum mechanical and classical measurements are derived statistically, typically per Zurek's program [23]. So, for example, spins input at angle θ to the apparatus would yield an average value of measured results given by $\cos\theta$, despite that the two eigenvalues are +1 and -1 and the apparatus can only give one of these two answers *no matter which way it points*. Susskind notes that this $\langle\sigma_n\rangle = \cos\theta$ is the same result we would get for a simple 3-vector in classical physics, and remarks [24]:

> *"Does our mathematical framework get the same result? <u>It had better!</u> If a theory disagrees with experiment, it's the theory that has to leave town."*

He claims $\langle\sigma_n\rangle = \cos\theta$ "*agrees perfectly with experiment.*" But *which* experiment does it agree perfectly with? This QM average or expectation value, $\cos\theta$, will be obtained by averaging the $+1$'s and -1's that constitute the QM-SG measurement data. That is,

$$\langle\sigma_n\rangle = \cos\theta \quad = \sum_i (+1)_i + \sum_j (-1)_j \,. \qquad 73$$

Our classical spin model produces an average $\cos\theta$ measurement containing the sum of *individual* $\cos\theta$–based measurements [equation (17) or asymmetric extensions (66) and (67)]. For angles $\theta < \pi/2$ all such measurements will be positive, in contrast to the quantum mechanical expectation, which allows only $+1$ and -1 measurement values to be used. So quantum mechanical *calculations* produce the correct expectation value, $\cos\theta$, but, to my knowledge, have never been *experimentally* verified. The atom-by-atom measurement of the proposed experiment will resolve this issue.

Our model of Stern-Gerlach and its analysis yields new physical results:

1. A new expression for the *finite decay time of precession*, and
2. *A threshold magnetic field gradient* below which SG will not work,
3. An energy-exchange analysis with both *free-energy* and *work* phases.

The model also implies results that are incompatible with current interpretations of quantum mechanics. This suggests that the 1922 Stern-Gerlach experiment, which serves as a fundamental prototype for quantum mechanics, should be repeated with improved technology and technique in order to answer several important questions.

It is difficult, post-Bell, for physicists to give serious consideration to a classical model of spin. The general belief is that Bell's theorem, and supporting experiments, prove classical models impossible. If that is true our proposed experiment should confirm it. Those considering our model seriously might bear two things in mind: first, actual Bell tests are performed with *photons*, not *spin-½ fermions*, and measurement differences are significant — *photons are counted*, whereas Stern-Gerlach measures *deflection*. Counting is an integrating procedure that can hide certain variables, whereas measure of deflection highlights the variable. Second, discussing his model, Bell effectively uses a force corresponding to $F = F\cos\theta / |\cos\theta|$. In other words he *removes* the classical θ -dependence from his QM model. Removal of θ physics from the model corresponds to conversion from a *'quantum physics'* problem to a *'quantum information'* problem, which is the focus of most papers on Bell's theorem in the literature.

Bell claims [4] p.145 that "*Certainly something must be modified [in the naïve classical picture] to reproduce the quantum phenomenon.*" The basic assumption (Susskind [24],p.71) is that "*For the familiar case of the spin, the possible values of any of the components are* ± 1. *The apparatus never gives any other result.*" This is based on the 'beam splitting' of the Stern-Gerlach experiment and the apparent decision to ignore the spread of data shown on the iconic post card as thermal in origin. Our experiment tightly filters the velocities of the particles, thereby constraining the '*thermal variation*', and individual initial angles are carefully prepared and individual particle deflections are measured, potentially challenging the basic quantum picture of Stern-Gerlach. After almost a century of confusion deriving from the usual quantum interpretation, it is believed that resolution of the basic question of spin dynamics is important.

The *Quantum Credo* contends that quantum mechanical *reality* yields the classical world of experience only in a statistical sense. A successful experiment would imply that *the quantum model of spin* is a statistical approximation to the reality described by the classical model. Other, non-spin, aspects of particles will be treated elsewhere.

I am indebted to Otis Lamont Frost for extensive discussion on the issues treated in this paper [29] .

Appendix

Shirokov [28] uses the Stern-Gerlach model Hamiltonian with the field gated *on* at $t=0$ and gated *off* at $t=\tau$ by $g(t)$

$$H=\frac{p^2}{2m}+g(t)\,\mu\vec{s}\cdot\vec{B}(\vec{x})\equiv H_0+H_1$$

where the inhomogeneous field $\vec{B}(\vec{x})$ is assumed linear

$$\vec{B}(\vec{x})=B_0+\vec{b}(\vec{x})$$

and $\vec{B}(\vec{x})$ must satisfy $\vec{\nabla}\cdot\vec{B}=0$ and $\vec{\nabla}\times\vec{B}=0$. He then solves the Schrödinger equation

$$i\partial_t\,\psi(t)=(H_0+H_1)\psi(t)$$

where $\psi_0=\phi_0\chi_0$ with the spatial part $\phi_0(\vec{x})$ being a wave packet and χ_0 being a spin wave function.

Shirokov shows that "expectation values of the operators p_i, $p_i p_j$, $p_i p_j p_k$ in the state $\psi(t)$ allow one to determine expectation values of the spin operator s_i and their products in the initial states, which is equivalent to the initial spin state determination. He remarks that state determination differs from the well-known quantum observable measurement.

The Stern-Gerlach device "*reduces measurement of spin observables to that of the particle position or momentum, the latter measurement being assumed possible*." For his purpose "*we must be able to measure momentum distributions which the particles had before and after the action of the magnetic field of the device*." Moreover, he claims that the usual treatment of the Stern-Gerlach device is based upon the entanglement of the spin and spatial parts of the particle wave function, but that Busch and Schroeck have shown that this $|\,\vec{p}\,\rangle\,|\,\vec{s}\,\rangle$ entanglement is only approximate.

In his approach the expectation values can be evaluated if the momentum distribution $w(p_1,p_2,p_3)$ is measured. Shirokov does not specify exactly <u>how</u> the momentum distribution is to be measured. Nevertheless with the Heisenberg operator $O(t)$, expressed to any order of the interaction H_i and with the usual commutation relations

$$[s_i,s_j]=i\,\varepsilon_{ijk}\,s_k \quad \text{and} \quad [\,p_i,f(x)]=i\frac{\partial f(x)}{\partial x_i},$$

he obtains (18) a lengthy expression for the Heisenberg operator p_k for $t>\tau$

$$\langle p_k \rangle = \langle p_k \rangle_0 - \mu \beta_k \tau T_k$$

$$+ \mu^2 \beta_k \int_0^t dt_1\, g(t_1) \int_0^{t_1} dt_2\, g(t_2) [\vec{T} \_ \ \langle \phi_0 | \vec{B}(t_2) | \phi_0 \rangle\ ]_k$$

$$- \mu^3 \beta_k \int_0^t dt_1\, g(t_1) \int_0^{t_1} dt_2\, g(t_2) \int_0^{t_3} dt_3\, g(t_3)$$

$$- \Bigg\{ \frac{1}{2} [[\vec{T} \_ \ \ \langle \phi_0 | \vec{B}(t_3)] \_ \ \ \vec{B}(t_2)]_k | \phi_0 \rangle$$

$$+ \frac{1}{2} [\ \langle \phi_0 | \vec{B}(t_2) \_ [\ \vec{B}(t_3) | \phi_0 \rangle \_ \vec{T}]_k$$

$$+ \frac{t_2 - t_3}{2m} \sum_{mn} \varepsilon_{kmn} \beta_n^2 \langle s_m s_n + s_n s_m \rangle - \Bigg\} + \cdots$$

Subtracting $\langle p_k \rangle_0$ from both sides yields the change $\langle p_k \rangle - \langle p_k \rangle_0$ of the momentum expectation value induced by the magnetic field, and he states that, *"if we hope, as usual, that the series (18) converges…"* then we will be able to obtain a simple expression for $\vec{T}$, the polarization tensor representing the initial spin state to order μ:

$$T_k = -\,[\langle p_k \rangle - \langle p_k \rangle_0\,] / \mu \beta_k\, \tau$$

We can now compare this quantum result to our energy-exchange approach. If we assume that the field β_k is in the x-direction $(k=1)$ with initial velocity in the z-direction $(k=3)$, then we expect no change in p_2 or p_3 hence

$$\langle p_k \rangle - \langle p_k \rangle_0 = m v_x - m v_{x0} \Rightarrow \quad m v_x$$

and

$$T_x = \frac{m v_x}{\mu B_x \tau}.$$

Multiply both sides by v_x and by τ to obtain $T_x v_x \tau = \frac{m v_x^2}{\mu B_x}$. If we let $x = v_x \tau$ be the vertical displacement in the field then we compare the result to our eqn (7):

$$\frac{m v_x^2}{2} = \mu B_x\,(x T_x) \qquad \Leftrightarrow \qquad \frac{m v_x^2}{2} = \mu B_x\,(1 - \cos\theta)$$

Shirokov's polarization tensor is a generalization of the polarization vector, and his quantum result appears not unrelated to our energy-exchange result. His is a generalized distribution tensor, whereas ours is the prediction of a trajectory, both based on the initial polarization state of the incoming spin.

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